On systematic serial concatenation of repeat and tree codes

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Abstract: *In this paper we have studied parallel concatenated tree codes and serial concatenation of repeat and tree codes for small to medium frame sizes. We have first analyzed the input redundancy weight enumerator function of these two concatenated codes using the probabilistic uniform interleaver. We have shown that they are similar to some extent. Considering both regular and irregular schemes, we have evaluated the performance degradation due to the iterative decoding. Finally we have proposed an interleaver construction in order to improve the performance of the codes.*

Keywords: turbo-like codes, repeat-accumulate codes, iterative decoding.

1. Introduction

Since the introduction of the parallel concatenated convolutional codes or turbo codes by Berrou et *als* [1], many related concatenated codes have been proposed or rediscovered such as regular and irregular LDPC [2], and repeat accumulate (RA) codes [3] [4]. In [5], Li Ping and *als* have proposed the so-called parallel concatenated tree codes (PCTC) composed of J 2 states recursive systematic convolutional coders and $J-1$ interleavers. In this paper, we will study the systematic serial concatenation of repeat and tree codes (SRTC) and will compare them to PCTC. We will show that this structure, simpler than the turbo codes, can achieve a good compromise between performance and complexity. We will first compare the Tanner graph of PCTC and CTC codes. Then, we will study their average performance using the probabilistic uniform interleaver. Both regular and irregular repeat codes will be considered. Finally, we will consider the construction of the interleaver.

2. Tanner graph of PCTC and SRTC codes

In the rest of this paper, a rate $\frac{r+s}{r+s+1}$ 2 states recursive systematic convolutional coder will be called (T, r, s) coder . r and s are respectively the number of bits included and excluded in the recursion. As shown on figure 1, the Tanner graph of these codes is a tree and can be decoded easily using the belief propagation algorithm.

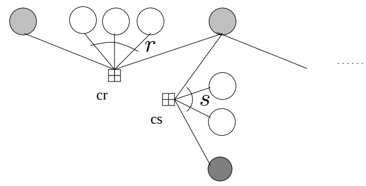

Figure 1: Tanner graph of a (T, r, s) coder.

The so-called accumulated codes are a special case of tree codes with $s = 0$.

Using Tanner graph we can easily describe the codes PCTC and SRTC. We show in figure 2 and 3 respectively the Tanner graph of the PCTC and SRTC codes with rate 1/2.

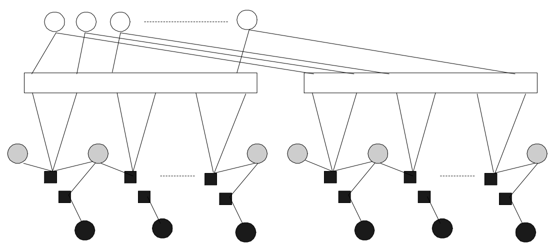

Figure 2: Tanner graph of PCTC codes with $J = 2$ $(T, 2, 1)$ codes

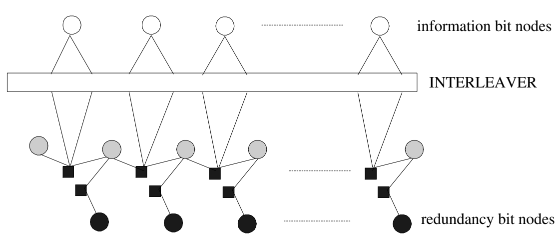

Figure 3: Tanner graph of SRTC codes with $J = 2$ $(T, 2, 1)$ codes

From these Tanner graphs we can see that there is only one constituent code and one interleaver for the

SRTC. On the other hand, for the PCTC the constituent code and the interleaver are divided into J parts (the first interleaver usually performs no interleaving). The first one is a serial concatenated code whereas the second is a parallel concatenated code. Since the accumulated codes are $(T, r, 0)$ tree codes, it is obvious that systematic RA codes belong to the family of SRTC.

3. Performance analysis of PCTC and SRTC codes

If we consider a (N, K) linear block code C with minimum distance d_{free}, the union bound on the bit error probability (BER) for maximum likelihood decoding of the code over additive white Gaussian channel is in the form :

$$Pb \leq \sum_{d=d_{free}}^{N} A_d H^d \mid_{H=e^{-R_c E_b/N_0}} \quad (1)$$

with

$$A_d = \sum_{d=w+z} \frac{w}{K} B^C_{w,z} = \sum_d \frac{w}{K} A^C_{w,d} \quad (2)$$

where $B^C_{w,z}$ called the input redundancy weight coefficient (IRWC) is the number of codewords in C with input weight w and redundancy weight z and $A^C_{w,d}$ called the input output weight coefficient (IOWC) is the number of codewords in C with input weight w and output weight d. Since the union bound is inaccurate in the region below the cutoff rate, we will use the tangential sphere (TS) bound proposed by Poltyrev [6] to evaluate the performances of the concatenated codes.

In [7] a probabilistic interleaver called uniform interleaver has been introduced. This interleaver is defined as a probabilistic device that maps an input sequence of weight w into all distinct permutations $\binom{N_i}{w}$ of it with the same probability $p = 1/\binom{N_i}{w}$ considering all the possible interleavers. Considering that all the interleavers are uniform, we can calculate the average input redundancy weight enumerator function (IRWEF) of the code from the IRWEF of the different constituent codes. For the multiple PCCC and the CTC composed of J identical constituent codes and $J-1$ interleavers, the IRWEF is given as follows:

$$B^{CP}(w, Z) = B^C(w, Z) \left[\frac{B^C(w, Z)}{\binom{K}{w}} \right]^{J-1} \quad (3)$$

where $B^C(w, Z) = \sum_z B_{w,z} Z^z$

The input output weight enumerator function (IOWEF) of a repeat accumulated code is obtained using the relation:

$$A^{RA}(w, D) = \sum_{l=0}^{JK} \frac{A^{CO}_{w,l} A^{CI}(l, D)}{\binom{JK}{l}} \quad (4)$$

with $A^C(w, D) = \sum_d A_{w,d} D^d$. A^{CO} and A^{CI} are respectively related to the repeat J time code and of the accumulate code.

Finally, for the SRTC codes the IRWEF is given as follows:

$$B^{SR}(w, Z) = \sum_{l=0}^{JK} \frac{A^{CO}_{w,l} B^{CI}(l, Z)}{\binom{JK}{l}} \quad (5)$$

In figure 4, we show the average BER of different PCTC and SRTC codes with rate 1/2, $J = 3$ and 4 and frame length $K = 440$ bits. When the constituent codes are $(T, 4, 0)$, we can observe that the waterfall region is almost the same for both schemes. On the other hand, the error floor is better for the PCTC code. This difference stems from the fact that for PCTC, the weight $w = 1$ input information words have no impact on the average BER whereas for SRTC, depending on the interleaving they could generate very low weight codewords. When these associations are avoided, the average BER of PCTC and SRTC are close.

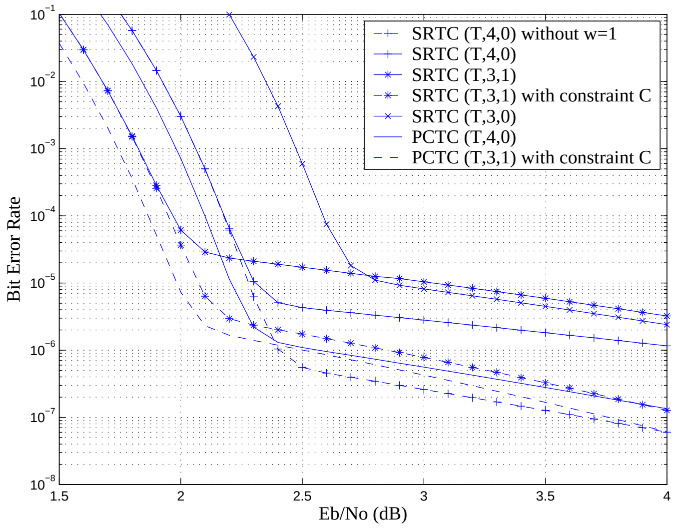

Figure 4: comparison of PCTC and SRTC TS bound $BER = f(E_b/N_0)$

We will restrict ourselves to $(T, r, 0)$ or $(T, r, 1)$ constituent coders since they give the best performance.

For both regular PCTC and SRTC, when using $(T, r, 1)$ tree codes, the interleavers should be built in order to impose the same protection to all the information bits. This restriction avoids low weight codewords such as $d = 1 + J$ codewords.

If $\Pi = [\pi_0, \pi_1, \ldots, \pi_{JK-1}]$ is the interleaving function and $u_i = v_{\pi_i}$, the constraint for the SRTC interleaver is :

$$\pi_i \mod (J) = i \mod (J) \qquad (6)$$

A similar constraint can be derived for the PCTC.

Except for a shift due to the small length of the frame, the E_b/N_0 corresponding to the waterfall regions are in accordance with the calculated asymptotical thresholds of these concatenated codes. The thresholds obtained using the gaussian approximation are 1.22 dB, 1.19 dB and 0.96 dB for the PCTC and SRTC with respectively $(T, 3, 0)$, $(T, 4, 0)$ and $(T, 4, 1)$ constituent codes [8].

4. Iterative decoding of regular and irregular SRTC

In this section we evaluate the performance degradation of the iterative decoding of regular and irregular SRTC compared to the ML decoding predicted by the TS bounds. For the design of the irregular SRTC with good degree sequences we have used the optimization method given in [4]. This method is based on the gaussian approximation and allows the construction of codes almost achieving the AWGN channel capacity.

In figure 5 we present simulated BER performances of two regular SRTC and irregular SRTC. The rate of these SRTC is fixed at 1/2 and the frame length is $K = 440$. The regular SRTC are composed of a J time repeat code and a $(T, J, 0)$ tree code with respectively $J = 4$ and 8. The first irregular SRTC is composed of an irregular repeat code with an average repetition factor equal to 4 and a $(T, 4, 0)$ tree code. The degree sequence of the irregular repeat code is $\lambda(x) = \sum_i \lambda_i x^{i-1}$ with $\lambda_2 = 0.13051$, $\lambda_3 = 0.2369$ and $\lambda_6 = 0.63258$. λ_i is the fraction of edges between the information nodes and the check nodes that are adjacent to an information node of degree i. The second irregular SRTC is composed of an irregular repeat code with an average repetition factor equal to 8 and a $(T, 8, 0)$ tree code. We have $\lambda_3 = 0.2527$, $\lambda_{11} = 0.0814$, $\lambda_{12} = 0.3271$, $\lambda_{46} = 0.1845$ and $\lambda_{48} = 0.1540$ [4]. The threshold for these 2 irregular SRTC is respectively 0.718 dB and 0.344 dB. All the simulations are performed using an iterative MAP decoder and 20 iterations. The interleavers have been chosen randomly.

We can see the difference between the asymptotical threshold and the E_b/N_0 of the TS bound waterfall region. While for regular SRTC, the difference is limited (0.9 dB at BER=10^{-3} for the (T,4,0) code), for the irregular SRTC the degradation is much more significant (1.6 dB at BER=10^{-3} for the (T,8,0) code). Consequently, the E_b/N_0 associated to the waterfall region of the irregular SRTC can be higher than the regular one. Furthermore, we can observe another degradation due to the iterative decoding. For the regular construction, this degradation is about 0.1 dB at BER=10^{-4} whereas the degradation is about 0.6dB for the irregular scheme. Whereas irregular SRTC can almost reach the channel capacity for large frame length, for small to medium frame lengths we get no improvement compared to the regular schemes. The main reason for this fact is the presence of many small cycles in the Tanner graph and as a consequence the message passing all along the iterations is not efficient.

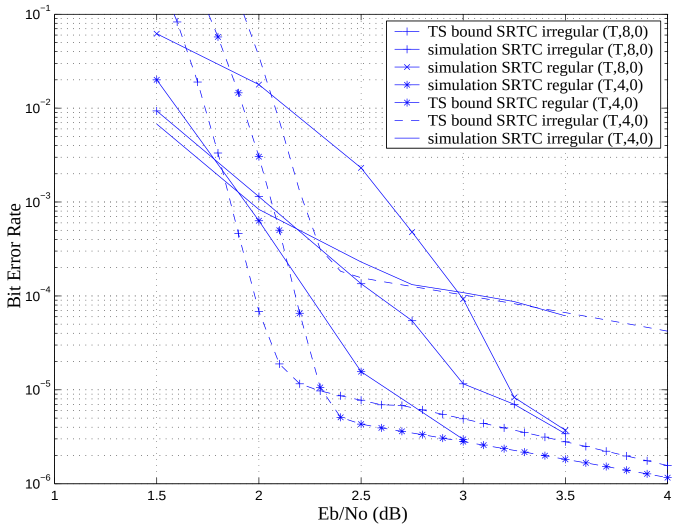

Figure 5: Performance $BER = f(E_b/N_0)$ of regular and irregular SRTC

5. Interleaver construction

When the frame sizes are small to medium, it is important to optimize the interleavers in order to improve both the weight distribution of the concatenated codes and the message passing since the decoding of these codes is performed iteratively. In this section, we will only consider the regular SRTC case.

For the $(T, r, 0)$ coder, the redundancy weight z associated to a weight 2 input sequence $u(D)$ with $u(D) = D^a + D^b$ is :

$$z \leq \left\lfloor \frac{|a - b|}{r} \right\rfloor + 1 \qquad (7)$$

For the $(T, r, 1)$ coder, we have also the same relation when both positions a and b are included in the recursion.

We can now give an upper bound on the minimum distance of a regular SRTC composed of $(T, r, 0)$ or $(T, r, 1)$ using the weight 2 input sequences. We define the dispersion factor L of an regular SRTC interleaver as follows :

$$L = \min_{i,j} \left(\sum_{n=1}^{J} (p(2n) - p(2n-1)) \right) \qquad (8)$$

where $p(2J) > p(2J-1) \cdots > p(1)$ are the $2J$ reordered interleaved positions associated to the $2J$ input positions $Ji, Ji{+}1, \ldots, Ji{+}J, Jj, Jj{+}1, \ldots, Jj{+}J$. Then we have :

$$d_{min} \leq d_{2min} \leq 2 + \left\lfloor \frac{L}{r} \right\rfloor \qquad (9)$$

As a consequence, maximizing d_{2min} is equivalent to maximizing L. Depending on J, it could be also necessary to consider the weight $w = 3, 4...$ input sequenced but the problem becomes more difficult due to the multiplicity of these sequences.

In [10], we have introduced the so-called cycle optimized interleaver (COI) construction for the Turbo-codes based on the elimination of the short cycles. This construction is equivalent to the S-random interleaver construction [9] except that the forbidden windows are triangular instead of rectangular. In this paper, we have applied the COI construction for the optimization of the interleaver of the SRTC.

In figure 6 we show simulation results of SRTC composed of a $J = 4$ time repeat code and a $(T, 3, 1)$ tree code. The frame length is $K = 456$ and $K = 1024$. We present both the BER performances using a random and a COI interleaver. The error floor of the SRTC with the COI interleaver is almost two decades lower than the SRTC with the random interleaver. The dispersion factor of the COI interleaver of size 4096 is 54. Nevertheless the error floor is still present at BER=10^{-7} and further optimization needs to be performed.

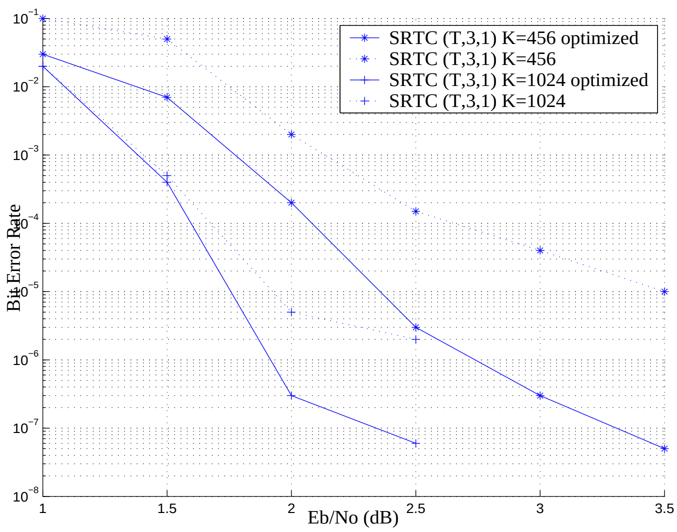

Figure 6: $BER = f(E_b/N_0)$ for SRTC with $K = 456$ and $K = 1024$

6. Conclusion

In this paper we have studied PCTC and SRTC. We have shown that these two families are very close. For small to medium frame lengths, we have also observed that the iterative decoding of irregular SRTC is less efficient than for the regular SRTC. Consequently, the performance of irregular SRTC are lower than the regular one. Finally we have proposed an interleaver construction in order to reduce the error floor of these codes. Even if the performance of the Turbo-codes still slightly outperforms the SRTC codes, the SRTC codes can achieve a very good compromise between performance and complexity.

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